Week 1 Group Problems — Topology — Spring 2023

1. Let's explore equivalence relations.

   (a) Take the set $\mathbb{R}^2 - \{0\}$, and declare $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $\lambda \in \mathbb{R}$ such that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$. Describe the equivalence classes. Can you think of ways to describe the set of equivalence classes?

   (b) On the same set, let $(x_1, y_1) \sim (x_2, y_2)$ if $\sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$. Describe each equivalence class, and describe the set of equivalence classes. Compare and contrast with part (a).

   Skip the next two and come back to them later.

   (c) Show that the intersection of a collection of equivalence relations is again an equivalence relation.

   (d) Intersect every equivalence relation on $\mathbb{R}$ containing $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x + 1 \text{ and } -1 < x < 1\}$. What do you get?

2. If X is a set, we let $i_X : X \to X$ denote its *identity function*. If $f : X \to Y$ then $g : Y \to X$ is a *left inverse* if $g \circ f = i_X$, and a *right inverse* if $f \circ h = i_Y$. Find f which has:

   (a) no left or right inverse (b) a left but not right inverse (c) a right but not left inverse

   (d) Can a function have more than one left inverse? right inverse?

   (e) Prove or disprove: a function $f : X \to Y$ can have left inverse g and right inverse h with $g \neq h$.

3. (a) Prove or disprove the existence of a strictly decreasing sequence of subsets of $\mathbb{R}^2$:

$$U_1 \supsetneq U_2 \supsetneq U_3 \supsetneq \cdots$$

(we write $A \supsetneq B$ if $A \supset B$ and $A \neq B$) such that if we let $U = \bigcap_{i=1}^{\infty} U_i$, then

(i) U is empty (ii) $|U| = 1$ (iii) $|U| = n$, for $n > 0$

(iv) U is countable (v) U is uncountable

(b) Repeat part (a) but with $\mathbb{Z}$ instead of $\mathbb{R}^2$.

(c) Challenge: Do (iv) for $(0, 1)$ instead of $\mathbb{R}^2$

4. I give you a real number $\epsilon > 0$. Give me a collection of closed intervals $\{[a_i, b_i]\}_{i \in I}$ (indexed by whatever set I that you want) inside $\mathbb{R}$ such that

$$\mathbb{Q} \subseteq \bigcup_{i \in I} [a_i, b_i]$$

and such that the sum of the lengths is $< \epsilon$:

$$\sum_{i \in I} (b_i - a_i) < \epsilon.$$

One way to interpret this statement is by saying that "$\mathbb{Q}$ has measure zero". It's "very small"!