Week 12 Group Problems — Topology — Spring 2023

Recall that I mentioned in class that $\pi_3(S^2) = \mathbb{Z}$, generated by something called the "Hopf fibration". This should be a map $S^3 \to S^2$.

Recall also that

$$S^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1 \right\}.$$

Most of the following problems are going to be algebra problems.

1. Show that

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ac + bd), 2(bc - ad))$$

   defines a map $S^3 \to S^2$. This is the *Hopf map*.

2. We can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by sending $(a, b, c, d) \mapsto (a + bi, c + di)$; how would you describe the image of S^3 under this identification?

3. We can identify S^2 with the one-point compactification of the complex numbers, written $\mathbb{C} \cup \infty$ (this is also called the complex projective line), via stereographic projection from the *unit sphere* to the $x - y$ plane. Using this (and the previous exercise), show that the Hopf map is the same as the map

$$(z_1, z_2) \mapsto z_1/z_2.$$

(Hint: if you can't remember how stereographic projection works, try it for the circle and $\mathbb{R} \cup \infty$ first)

4. Using the previous exercise, compute the fibers of the Hopf map over a fixed point.

5. Assuming you know that the Hopf map is a fiber bundle, show that it can't be the trivial fiber bundle.

6. Go watch this video: https://www.youtube.com/watch?v=AKotMPGFJYk. In this video S^2 is viewed in the usual way, and S^3 is viewed as $[0, 1]^3$ with the boundary glued together.