1 Open/closed sets

We saw a notion of open and closed ball; $B_r(x)$ and $B_r^\bullet(x)$ in a metric space. Let's expand this a bit.

1. If (M, d) is a metric space and $U \subseteq M$ is a subset of M, say U is *open* if for every point $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$.
   (a) Is the empty subset open?
   (b) Is an arbitrary union of open subsets open? Prove or find a counterexample/salvage.
   (c) Is an arbitrary intersection of open subsets open? Prove or find a counterexample/salvage.
   (d) If M is discrete, what are the open sets?
   Skip the next problem and come back to it later.
   (e) Can you precisely describe the structure of *every* open set in $\mathbb{R}$ (with the Euclidean topology)?

2. If (M, d) is a metric space and $Z \subseteq M$ is a subset, we say that Z is *closed* if its complement in M is open.
   (a) Is it true that if $Z \subseteq M$ is closed and $x \in Z$ then Z must contain a closed ball $B_r^\bullet(x)$?
   (b) Is an arbitrary union of closed subsets closed? Prove or find a counterexample/salvage.
   (c) Is an arbitrary intersection of closed subsets closed? Prove or find a counterexample/salvage.
   (d) If M is discrete what are the closed sets?
   (e) Can you find a subset of a metric space which is neither closed nor open?
   (f) Can you find a subset of a metric space which is both closed *and* open (clopen)? What about in $\mathbb{R}$?
   (g) Show that $\{x\} \subset M$ is always closed.
   Skip the next problem and come back to it later.
   (h) Prove that a subset $Z \subseteq M$ is closed if and only if
   $$(x_i) \in Z \text{ and } x_i \to x \in Z \text{ implies } x \in Z.$$
   (hint: think about $\mathbb{R}$ or $\mathbb{R}^2$ first to get some intuition!)

2 Ultrametrics

A metric d on a space M is called an *ultrametric* if it satisfies the *strong triangle inequality*

$$d(x,z) \leq \max(d(x,y), d(y,z))$$

1. Let $\mathbb{C}((x))$ denote the set (actually, it's a field, but we don't need this) of *Laurent series* in the complex numbers. This is the set of power series

   $$f(x) = \sum_{i=N}^{\infty} c_i x^i$$

   with $c_i \in \mathbb{C}$ and for $N \in \mathbb{Z}$, *positive or negative*; so in particular $\{i < 0 : c_i \neq 0\}$ is finite. If $f(x) \in \mathbb{C}((x))$, let $|f(x)| = 2^{-i}$ where i is the smallest index for which $c_i \neq 0$ (e.g. if $f(x) = x^{-1} + 2x$ then $i = -1$). Define a metric on $\mathbb{C}((x))$ by taking

   $$d(f(x), g(x)) = |f(x) - g(x)|.$$

   Show that d is an ultrametric.

2. Now let (M, d) be an arbitrary ultrametric space. These spaces are really weird! Here's why.

   (a) In the inequality

   $$d(x,z) \leq \max(d(x,y), d(y,z))$$

   show that if $d(x,y) \neq d(y,z)$, then $d(x,z) = \max(d(x,y), d(y,z))$.

   Give an example in $\mathbb{C}((t))$ where the inequality is strict.

   (b) Show that if $y \in B_r(x)$ then $B_r(x) = B_r(y)$. Conclude that if $B_r(x) \cap B_s(y) \neq \varnothing$ then one is contained in the other.

   (c) Show that every closed ball is open and that every open ball is closed.

3. Can you draw a picture of an ultrametric space?