Week 3 Group Problems — Topology — Spring 2023

Here are some warm-up problems about convergent sequences.

1. Give an example of a sequence of real numbers (a_n) which has no convergent subsequence

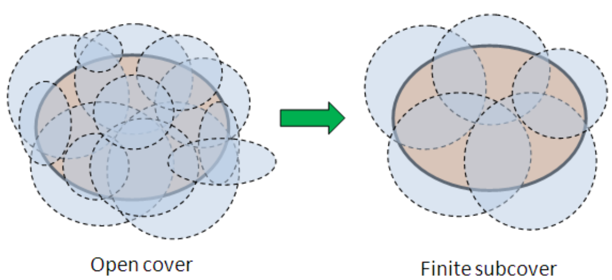
2. Give an example of a sequence of *integers* (a_n) such that for every integer $N \in \mathbb{Z}$, (a_n) has a subsequence that converges to N.

3. Give an example of a sequence of real numbers (a_n) such that for every $r \in \mathbb{R}$, (a_n) has a subsequence that converges to r.

Definition: If M is a metric space and $\{U_i\}_{i\in I}$ is an arbitrary collection of open subsets of M such that $\bigcup_{i\in I} U_i = M$, then we say that $\{U_i\}_{i\in I}$ is an *open cover* of M.

Definition: We say that M is *compact* if every open cover $\{U_i\}_{i\in I}$ of M has a finite subcover, meaning that there exist $i_1, \dots, i_n \in I$ such that

$$M = U_{i_1} \cup \cdots \cup U_{i_n}.$$



Theorem: A metric space is compact if and only if it is sequentially compact.

Goal: Explore compactness and prove (one direction of) the theorem!

1. As usual, let's start with the discrete space. When is a discrete metric space compact? Prove it.

2. Consider the open disk

$$B_1((0,0)) = \left\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\right\}$$

   Find an open cover of $B_1((0,0))$ which has *no* finite subcover (so it's not compact). Can you choose this open cover to consist of disjoint open subsets or not? Why or why not?

3. We saw that in Euclidean space, sequential compactness is the same as closedness and boundedness. So we know that the closed ball of radius 1 $B_1^\bullet((0,0)) \subseteq \mathbb{R}^2$ is compact. Try to give an open cover that has no finite subcover (you can't do it; but briefly give it a try to appreciate why not!)

4. Now show that if M is compact, then it is sequentially compact (hint: try to find a contradiction). We won't prove the converse now, but talk to me if you are interested in trying.

5. If M is a metric space and $A, B \subseteq M$ are two subsets, define

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$$

(remember that inf means "greatest lower bound") Show that if A and B are compact then $d(A, B) = d(p, q)$ for some $p \in A$ and $q \in B$. Some tips:

(a) Draw a few examples of random compact sets A and B in $\mathbb{R}^2$ and convince yourself this is true.

(b) Find a counterexample if one of the sets are not compact to see why you need compactness.