Week 4 Group Problems — Topology — Spring 2023

A topological space is defined by specifying which sets are open, or equivalently, which sets are closed. Today we will explore some other ways to define a topology.

Definition 0.0.1. Fix X a topological space and $A \subseteq X$ a subset.

1. The *closure of* A, denoted $\overline{A}$, is the smallest closed subset containing A; in other words, $\overline{A}$ is the intersection of all closed subsets containing A (why is this closed?).
2. The *interior of* A, denoted A°, is the largest open subset contained in A; in other words, A° is the union of all open subsets contained in A (why is this open?).

On the homework you will show that a subset $Z \subseteq X$ is closed if and only if it contains all of its *limit points*, so you can also define $\overline{A}$ to be the set of all limit points of A.

1. In Euclidean space, what is the closure of an open ball of positive radius? What is the interior of a closed ball of positive radius?

2. In $\mathbb{R}$ what is the interior of a single point? In $\mathbb{R}^2$ what is the interior of the x-axis?

3. In $\mathbb{R}$ what is the closure of $\mathbb{Q}$? What about the interior of $\mathbb{Q}$?

4. If X is a discrete topological space, describe what interior and closure do. Same for indiscrete spaces.

5. Show that $\overline{\overline{A}} = \overline{A}$. Show that $(A^\circ)^\circ = A^\circ$.

6. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Show that $\overline{\varnothing} = \varnothing$.

Now that we've seen some basic properties of closure and interior, let's try to reverse things. So now forget you have a topology at all, and forget you have open and closed sets entirely.

Definition 0.0.2 (Kuratowski closure axioms)**.** If X is a set, then a *closure operator* is a map $\overline{(\cdot)} : \mathcal{P}(X) \to \mathcal{P}(X)$ (this notation means that $A \mapsto \overline{A}$ under this map, for $A \in \mathcal{P}(X)$) satisfying

1. $\overline{\varnothing} = \varnothing$,
2. $A \subseteq \overline{A}$ for all $A \in \mathcal{P}(X)$,
3. $\overline{\overline{A}} = \overline{A}$ for all $A \in \mathcal{P}(X)$, and
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Now suppose you have a closure operator on X. We say that a subset $Z \subseteq X$ is *closed* if $\overline{Z} = Z$.

1. Show that the closed sets, defined this way, define a topology. In other words, show that $\varnothing$ and X are closed and that closedness is stable under finite union and arbitrary intersection.

2. Ty to formulate a dual notion of "interior operator". Define a notion of "open set" with respect to your definition, and show that the collection of open sets with respect to an interior operator forms a topology.

Bonus exercise: show that a function $f : X \to Y$ between two topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.