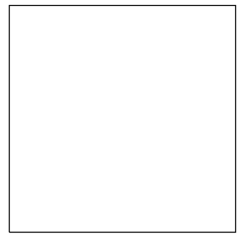
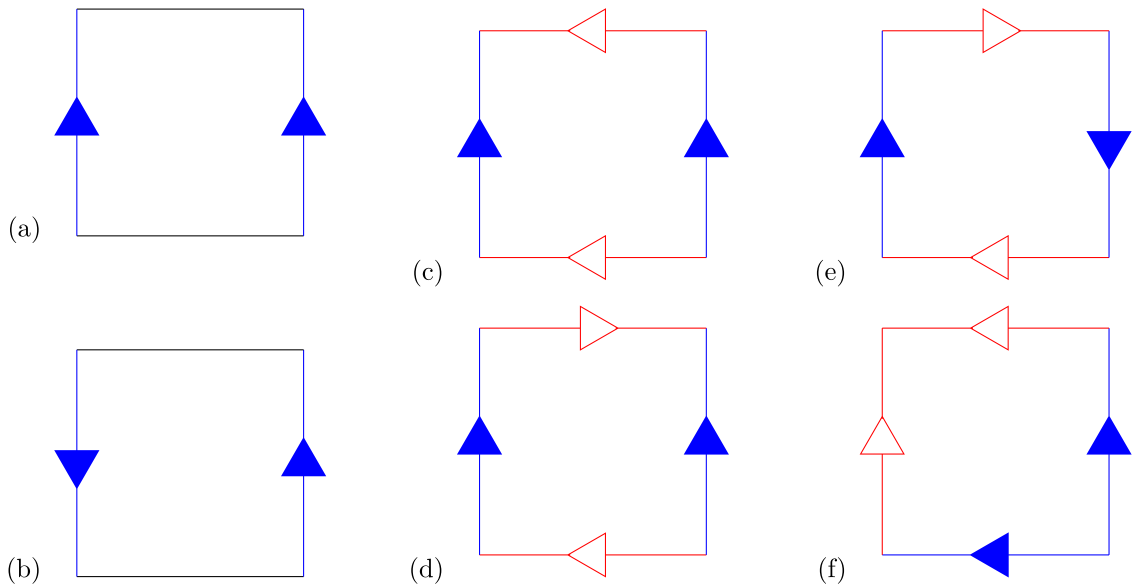
Today we'll explore product, quotient, and subspace topologies.

1. (Quotient topology exercise) Consider a square piece of paper.



Let's try gluing it to itself in different ways. Glue the edges of the pieces of paper with the same color/shading type, making sure that the triangles line up with each other properly (i.e. not upside-down from each other). What shapes do you get?



2. Recall that a *group* is a set G with a binary operator $- \bullet - : G \times G \to G$, an *identity* element $e \in G$, and an *inverse map* $i : G \to G$ satisfying

   (a) (associativity) $(g \bullet h) \bullet k = g \bullet (h \bullet k)$ for all $g, h, k \in G$

   (b) (two-sided identity) $g \bullet e = e \bullet g = g$ for all $g \in G$

   (c) (inversion) $g \bullet i(g) = i(g) \bullet g = e$ for all $g \in G$.

   Let's spice this up a bit. Pick a topology on a group G. With this choice of topology, say that G is a *topological group* if

   (a) The map $- \bullet - : G \times G \to G$ is continuous. Here we put the product topology on $G \times G$.

   (b) The map $i : G \to G$ is continuous.

   Now let's do some examples.

   (a) Observe that every group is a topological group, if you give it the discrete topology.

   (b) Show that $\mathbb{R}$, viewed as a group under addition and equipped with the Euclidean topology, is a topological group.

   (c) Show that the unit circle $S^1 \subset \mathbb{C}$ inside the complex plane is a topological group under multiplication; here we have the Euclidean topology again.

   (d) Show that $\mathbb{R}^\times$ (the units in $\mathbb{R}$, equivalently the nonzero elements of $\mathbb{R}$) is a topological group for multiplication, and the subspace topology of the Euclidean topology.

   (e) Let

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$$

   denote the set of 2×2 invertible matrices with coefficients in $\mathbb{R}$. This is a group under multiplication. Observe that there is an injective map

$$\begin{aligned} \mathrm{GL}_2(\mathbb{R}) &\hookrightarrow \mathbb{R}^4 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (a, b, c, d) \end{aligned}$$

   and so you can give $\mathrm{GL}_2(\mathbb{R})$ the subspace topology from $\mathbb{R}^4$. Show that $\mathrm{GL}_2(\mathbb{R})$ is a topological group. Is $\mathrm{GL}_2(\mathbb{R})$ open or closed in $\mathbb{R}^4$?