Week 7 Group Problems — Topology — Spring 2023

Not all spaces are compact; e.g. $\mathbb{R}$. But there are ways to embed it into a bigger space; for example, you can notice that $\mathbb{R} \cong (0,1)$ and then glue a point to both ends to make a circle.

Definition. A continuous map $f : X \to Y$ between two topological spaces is an *embedding* if it induces a homeomorphism $f : X \to f(X)$ onto its image (where $f(X)$ has the subspace topology from Y).

This definition excludes bijective continuous maps whose inverse is not continuous.

Here's the *one point compactification.* If X is a topological space, let $X^\star := X \sqcup \{\infty\}$. Define a topology on $X^\star$ where the open sets are

- open subsets of X itself, and
- sets of the form $(X - C) \cup \{\infty\}$ with $C \subseteq X$ a closed and compact subset.

1. Show that this defines a topology.

2. Show that the inclusion $X \to X^\star$ is an embedding, as defined above, and show that the image is open.

3. If X is already compact, then describe the topology on $X^\star$.

4. If X is not compact, show that the closure of X in $X^\star$ is all of $X^\star$ (when this happens we say that X is *dense* in $X^\star$). Is this true when X is compact?

5. (Skip this and do it later: turn to the back first) Show that $X^\star$ is Hausdorff if and only if X is Hausdorff and is *locally compact*, meaning that for every $x \in X$ there exists a compact set $K \subseteq X$ and an open set U satisfying $x \in U \subseteq K$.

6. What is the one point compactification of $\mathbb{R}^n$? Of $[0,1)$? Of $(0,1)\cup(2,3)$? Of $\{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$?

7. Let $\mathbb{N}$ denote the natural numbers with the discrete topology. Then a continuous function $\mathbb{N} \to \mathbb{R}$ is the same as a sequence. Describe what it means for a function $\mathbb{N}^\star \to \mathbb{R}$ to be continuous.

8. Bonus, slightly unrelated exercise: suppose X is a Hausdorff topological space and $R \subseteq X \times X$ is an equivalence relation. Show that $X/\sim$ with the quotient topology is Hausdorff if and only if $R \subseteq X \times X$ is closed, and find a counterexample when it is not closed.