We defined the fundamental group as the set of path-homotopy equivalence classes of loops based at a fixed point. Note that there are lots and lots of loops in general (for example, even in $\mathbb{R}^n$, but since a lot of them are path-homotopic to one another, there are "really not so many" up to path-homotopy.

But when you construct a path-homotopy, you end up constructing an *infinite family of loops*, and when you move along the homotopy, you are starting from one loop, going to a loop that is "nearby", etc. So it seems like the loops *themselves* should fit into some kind of space, and the path-connected components of this space should be in bijection with the fundamental group!

Let's try to define this space. The underlying set consists of continuous maps $(S^1, *) \to (X, x_0)$ (where $*$ denotes an arbitrary point on S^1 and $x_0 \in X$ is some point). What is the topology on it?

1. First, a warm-up. If X, Y, Z are three sets, describe a natural bijection

$$\Phi : \mathrm{Fun}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Fun}(X, \mathrm{Fun}(Y, Z))$$

2. Now suppose X, Y, Z are topological spaces. If $f \in \mathrm{Fun}(X \times Y, Z)$ is continuous, show that $\Phi(f)(x)$ (which is a function $Y \to Z$) is continuous for all $x \in X$.

Now let $C(X, Y)$ denote continuous maps from $X \to Y$. You showed that there is a map

$$C(X \times Y, Z) \to \mathrm{Fun}(X, C(Y, Z)).$$

You might then naturally ask: is there a topology one can put on $C(Y, Z)$ so that I can replace "Fun" with C in the map above and get a bijection? In other words, is there a topology so that $f : X \times Y \to Z$ is continuous if and only if $\Phi(f) : X \to C(Y, Z)$ is continuous?

Definition 0.0.1. If X, Y are two topological spaces, the *compact-open topology* on $C(X, Y)$ is the one generated by the sub-base of open sets consisting of

$$\{P(K, U)\}$$

where $K \subseteq X$ is compact, $U \subseteq Y$ is open, and

$$P(K, U) = \{f \in C(X, Y) : f(K) \subseteq U\}.$$

1. What is the topological space $C(*, Y)$?

2. If X is discrete, what is the topological space $C(X, Y)$?

3. If X, Y, Z are topological spaces and $f : X \to Y$ is continuous, show that

$$\begin{aligned} C(Z, X) &\to C(Z, Y) \\ g &\mapsto f \circ g \end{aligned} \qquad\qquad \begin{aligned} C(Y, Z) &\to C(X, Z) \\ g &\mapsto g \circ f \end{aligned}$$

are continuous.

4. If X, Y, Z are topological spaces and $f : X \times Y \to Z$ is continuous, show that the corresponding map

$$\Phi(f) : X \to C(Y, Z)$$

is continuous for the compact-open topology, and therefore gives a map

$$C(X \times Y, Z) \to C(X, C(Y, Z)).$$

This is not always a bijection! It's only true when X is locally compact and Hausdorff.

Definition 0.0.2. If (X, x_0) is a topological space with a fixed basepoint, its *loop space* is

$$\Omega X = \{\ell : I \to X : \ell(0) = \ell(1) = x_0\} \subset C(I, X)$$

equipped with the subspace topology from the compact-open topology.

Bonus exercise: using the fact that $C(X \times Y, Z) \xrightarrow{\sim} C(X, C(Y, Z))$ when X is locally compact Hausdorff, show that the set of connected components of ΩX is actually in bijection with $\pi_1(X, x_0)$.